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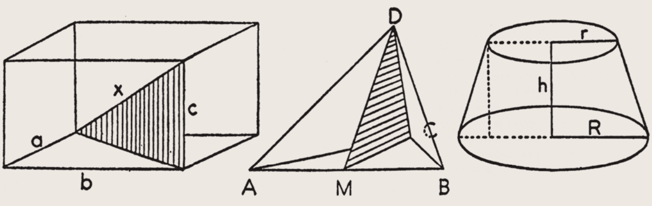
How to Solve It



A system of thinking to help you solve any problem

George Pólya

PENGUIN BOOKS

How to Solve It

George Pólya (1887–1985) was a Professor of Mathematics at Stanford University.

GEORGE PÓLYA

How to Solve It

A New Aspect of Mathematical Method

PENGUIN BOOKS

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Foreword

by Ian Stewart

During my last couple of years at school I used to haunt the public library, looking for mathematics books. I now realize that this was not a normal activity for a 16-year-old, but at the time it seemed entirely natural, for I was in the grip of an irresistible addiction: mathematics. I still am. Among the scores of books that I devoured was one that was, even then, a classic. You hold it in your hands at this moment: *How to Solve It.*

Any red-blooded mathematician would sign a pact with the Devil for that information. Mathematics is *hard.* So are most things that are worth doing, but mathematics demands an unusual mix of intellectual curiosity and nit-picking pedantry. George Pólya knew that mathematics is hard, but unlike most practitioners of the arcane art, he wanted to make it easier. He was a first-class research mathematician, a brilliant teacher and an able expositor. You won't find that combination often.

Pólya noticed that his students didn't know how to solve problems. Countless thousands of mathematics teachers have observed the same thing, but Pólya's thinking went a little deeper. The difficulty was not that his students didn't *know* enough mathematics, or that they didn't understand the mechanics of using what they knew. He came to the conclusion that what they lacked was the ability to direct their thought processes along fruitful channels. They might be able tacticians, but their sense of strategy was faulty. Could this be because they had no idea that there was any such thing as a strategy for solving a mathematical problem?

Pólya's experience in research led him to recognize that there are a number of general problem-solving techniques, which mathematicians use all the time but seldom articulate. He called them *heuristic strategies*. At the beginning of his book you'll find a skeleton outline, dividing the process into four phases:

- understand the problem
- try to use experience from related problems to plan an attack
- carry out the attack and, finally,
- ask yourself whether you really believe the answer you've got.

There are fashions in the teaching of mathematics. Problem-solving came into vogue in the 1980s, in part as a reaction against the abstraction of 'New Mathematics'. Pólya became the unwitting guru of the problem-solvers. The 1980 yearbook of the National Council of Teachers of Mathematics in the USA reads as if it has been marinated in Pólya sauce. More recently the idea has resurfaced as the buzzword *investigation*. Children should not learn facts or methods: they should investigate problems for themselves.

It sounds exciting. But what is the evidence? Can people really use Pólya's heuristic strategies to solve problems? On the face of it, the answer seems to be obvious. The anecdotal evidence from practising mathematicians is massive and convincing. *Yes, that's the way mathematicians think*. On the other hand . . . teachers who coached students for the International Mathematics Olympiads came to a unanimous conclusion. Students *don't* learn to solve problems by following Pólya's heuristic strategies. They learn to solve problems by starting with lots of raw talent and honing it razor-sharp on lots of abrasive problems. Programmers trying to develop Artificial Intelligence found that computers couldn't use heuristics either. The future of Pólya's brainchild began to look less bright.

By now you must be wondering whether I'm going to tell you not to buy the book. Not so. Didn't I just say it's a classic? But, in any case, coaches of would-be Olympians and the Artificial Intelligentsia notwithstanding, Pólya was *right*.

Provided you apply his strategies at the right level.

Let me leave that remark hanging, as one might a pheasant, to mature it, while I tell you a little about the man himself.

'MATHEMATICS IS IN BETWEEN . . .'

George Pólya was born in Budapest on 13 December 1887. As a child he did not find mathematics especially interesting: he recalled that of his mathematics teachers 'two were despicable and one was good'. He was very bright: his position in class at the Gymnasium, or secondary school, varied between second and fourth, and apparently he had no trouble maintaining it. At that time Hungary ran the only national mathematics competition in the world for secondary-school pupils, the Eötvös Competition. All students entering college were encouraged to take part. Pólya went to the test centre, but didn't hand in his paper.

In 1905 he began his studies at the University of Budapest. His mother insisted that he should study law. He stood the boredom for one term. He changed to languages and literature, and then to philosophy. As part of his philosophy course he was advised to take mathematics and physics. As a result he came into contact with two outstanding scientists: the physicist Loránd Eötvös, and the mathematician Lipót Fejér. Fejér's lectures were famous, and they attracted many into mathematics. He used to sit in cafés talking to his students about mathematical problems and telling them tales of famous mathematicians. Pólya was among those hooked. 'I thought, I am not good enough for physics and I am too good for philosophy. Mathematics is in between.'

In 1912 he gained his Ph.D., which was in mathematics with a minor in physics and chemistry. His thesis research was in probability theory. He did post-doctoral work at Göttingen and Paris, and in 1914 took up a teaching position at the Federal Institute of Technology in Zürich. At the outbreak of war he tried to join the Hungarian army but was rejected because of the after-effects of a childhood soccer injury. Later a more desperate Hungary tried to recall him from Switzerland, but by then Pólya had read Bertrand Russell and decided that war was wrong, and he stayed put. In 1918 he married Stella Weber, a Swiss.

In 1940, one World War later, along with thousands of other European intellectuals who found the activities of Adolf Hitler intolerable, the Pólyas arrived in the United States. After a two-year visiting position at Brown University, Pólya settled down in California at Palo Alto and took a post at Stanford University. Here, in 1945, he wrote *How to Solve It*. The book has since sold more than a million copies and has been translated into seventeen languages. He wrote three other books with an educational bent, and four research monographs. His collected papers fill four volumes. The Mathematical Association of America produced a film of his lectures, called *How to Teach Guessing*. It won the 'blue ribbon' in the Educational Film Library Association's Film Festival in 1968.

Pólya's research touched many fields of mathematics, among them complex function theory, combinatorics and probability theory. His classification of the seventeen discrete symmetry groups in the plane ('wallpaper patterns') had a significant influence on the artist Maurits Escher, who studied Pólya's paper carefully, transferring it in full to his notebooks.

He was conservative about fashion in mathematics and in education. When he was in Zürich there was a great deal of interest among mathematicians in 'intuitionistic logic', which holds that a proposition *P* and its double negative not-not-*P* may be different. Hermann Weyl, an enthusiast for intuitionism, bet Pólya that within fifty years the whole of mathematics would have been rewritten in intuitionistic terms; Pólya begged to differ. The terms of the bet were inscribed on a document, to be opened fifty years later. When it was, Pólya won hands down.

Many of Pólya's sayings have been preserved. Asked which mathematician had influenced him most, he said it was Leonhard Euler (Swiss, 1707–83): 'Euler did something that no other great mathematician of his stature did. He explained how he found his results, and I was deeply interested in that. It has to do with my interest in problem-solving.' He had many students, among them John von Neumann, one of the fathers of the electronic computer, a man so versatile that his involvement with ENIAC was almost a sideline. Pólya said that von Neumann was 'the only student of mine I was ever intimidated by. He was so quick. There was a seminar for advanced students in Zürich that I was teaching and von Neumann was in the class. I came to a certain theorem, and I said it is not proved and it may be difficult. Von Neumann didn't say anything but after five minutes he raised his hand. When I called on him he went to the blackboard and proceeded to write down the proof. After that I was afraid of von Neumann.'

Pólya remained interested in mathematics throughout his long life, but he felt his age keenly and often reminded visitors that he was approaching a full century. When computers started to make an impact on the teaching of mathematics, Agnes Wieschenberg discussed them with him. 'I am almost 100 years old, too old to learn computers, but if I would live in New York I would listen to your computer classes,' said Pólya. Paul Erdös promised him a 100th birthday celebration. He replied, 'Maybe 100, but not more.'

He died in Palo Alto on 7 September 1985, aged 97.

GUIDELINES, NOT RULES

Let me return to the question of heuristic strategies. Pólya saw them not as a rigid recipe but as a set of practical guidelines. It is inherent in the nature of guidelines that they don't work if you take them too literally. They are something that you must interpret through the eyes of experience. This explains at once why heuristics alone are of little use in Artificial Intelligence. But if they are embedded in a richer structure of machine inference, it turns out that they perform quite well. Pólya's strategies relate to a much deeper level than the operational surface of mathematics. In the same way, it looks as though the Olympians possessed so much raw talent that they already 'knew' the heuristic strategies – and a lot more. Their main problem was to enlarge the background against which those strategies could operate. Educationalists have found that Pólya's basic ideas can be made to work but that the skeleton which he laid down needs to be fleshed out before it leads to a successful teaching procedure. Each of Pólya's general strategies must be expanded into a group of related, but distinct, operational tactics.

For example, one principle is that a general problem can often be illuminated by considering special cases. But the type of special case will vary from problem to problem. In questions about the summation of n terms of a series, it is often worth working out the first few cases, $n = 1, 2, 3, \ldots$ If the problem is about the divisibility properties of integers, then the 'right' special case may be when n is prime or when n has a small number of factors, but this time it seldom helps to look at the cases $n = 1, 2, 3, \ldots$

Thus another educational problem arises, one that is dealt with implicitly in *How to Solve It* but not perhaps given the emphasis it deserves. If a dozen or so general strategies are replaced by several hundred tactics, how is the student to select which one to use? According to Alan Schoenfeld, 'Research now indicates that a large part of what comprises competent problem-solving behaviour consists of the ability to monitor and assess what one does while working problems, and to make the most of the problem-solving resources at one's disposal. It also indicates that students are pretty poor at this, partly because issues of "resource allocation during thinking" are almost never discussed.' In short, the would-be problem-solver needs to develop a feeling for when the attack is making progress or when it's bogged down in a dead end. The buzzword for this is *metacognition*.

Developing this kind of ability requires a mixture of general guidelines, specific methods, plenty of practice on examples and the encouragement of a certain kind of introspection. It's as much an art as a science.

PÓLYA'S FOUR PHASES

You may feel that the first phase in solving a problem by Pólya's method scarcely needs to be made explicit. *Understand the problem.* Obvious, isn't it?

Well, no. And it's a measure of the man's genius that he recognized the need for it. I never cease to be amazed at the number of perfectly intelligent students who have never appreciated this simple point for themselves. 'I can't do this problem about continuous functions.' 'OK, then, what does "continuous" mean?' (said rhetorically). 'Um – not sure.' 'How on earth do you expect to solve a problem when you haven't the foggiest idea what it means?' There are times when I could cheerfully strangle them.

The second phase, devising a plan of attack, is a much more complex part of the process. Problem-solving ability doesn't develop in a vacuum. It needs a rich background of knowledge and intuition before it can operate effectively. Pólya places a great deal of emphasis on the consideration of related problems whose solution is already known and on reasoning by analogy. Analogy can be a powerful weapon, but it takes broad experience to draw the *right* analogy.

About three years ago one of my colleagues at Warwick University, Mark Roberts, showed me a diagram that he had found in a research article, referring to a gadget called the Hénon-Heiles Hamiltonian. A Hamiltonian can be thought of as the energy function for a system that conserves energy. The Hénon-Heiles Hamiltonian occurs in a problem in dynamics that originally came from astronomy, and the aim is to understand the solutions to a particular, deceptively simple, differential equation. The diagram had been drawn on a computer, and among other things it showed that the equation has exactly eight solutions that are periodic, that is, lead to dynamic motions that repeat the same thing over and over again. Five of these solutions are stable and three are unstable.

Mark pointed out that exactly the same sort of behaviour occurs in a rather different problem, known as Hopf bifurcation, with which we both had some familiarity. Hopf bifurcation is about the onset of wobbles in systems whose steady states become unstable. The numerology of eight periodic states, five stable, three unstable, is exactly what you expect in Hopf bifurcation in any system with the same symmetry as an equilateral triangle. Moreover, the Hénon-Heiles Hamiltonian can be interpreted as the motion of a particle in a potential well that has precisely this symmetry. Solved it? Not at all – there was a snag. For technical but important reasons, periodic solutions in Hamiltonian dynamics do not arise through Hopf bifurcation.

Here our understanding had got bogged down at Pólya's final stage: checking the reasoning. The argument didn't hang together. So something was missing. It seems to me that many people – politicians spring to mind – have no real wish to understand the nature of a problem. They start with an answer and bend the evidence to fit it. With that approach we could just have mumbled a few incantations: 'Periodic solutions . . . symmetry . . . obvious.' But a critical look made it clear at once that the argument was full of holes.

None the less, the results couldn't possibly be coincidence, which meant we were back to phase two: planning an attack. Together with a third mathematician, James Montaldi, we started tracking down the reasons. Out of this one idea developed a programme to study Hamiltonian dynamics for symmetric systems, which has been going for three years and will in all likelihood continue for another five or more. It has turned out to be an incredibly rich area, with lots of potential applications to problems as varied as the vibrations of a methane molecule and the evolution of spiral galaxies. We're now working out all the bits and pieces, Pólya's third phase. In practice all the phases get mixed up and are carried out in parallel; for example, each new discovery tends to modify the overall plan.

All this from one picture. Inspired by an analogy, by family resemblances in two problems that we *knew* were quite different on a technical, operational level. This is not just problem-solving: it's problem-generation. The point that I want to make clear is that, whatever the role of Pólya-type heuristics, someone has to have the two pieces of the puzzle, Hénon-Heiles and Hopf, in his or her head at the same time, otherwise nobody will spot the possible connection. You can't consider lots of related but solved problems unless your head is full of all sorts of bits of mathematical reasoning.

There's an ironic twist to this particular tale. When we started working on the problem we didn't know much about Hamiltonian dynamics. Some time after we had gained a reasonable understanding of what was going on, and why the Hénon-Heiles system looked like a triangularly symmetric Hopf bifurcation, we were looking in a standard research text and found a simple trick, known to most people in Hamiltonian dynamics, that converts one into the other. Even though periodic solutions to Hamiltonian systems don't arise by Hopf bifurcation, you can still detect them that way: you merely have to tinker with them to make them non-Hamiltonian in just the right manner, by adding a bit of friction at the beginning and then taking it away at the end. If we'd known that to begin with, we would probably have decided that there wasn't anything really interesting involved. But by the time we found out, we'd already pushed the theory well beyond what you could get from Hopf bifurcation, by using quite different ideas to get results that won't come from the simple textbook trick. So in this case some selective ignorance turned out to be crucial. Ignorance is a strategy that, on the whole, I don't think you'll find in Pólya. Fortunately it's one that you don't have to teach: most of us can be pretty ignorant about almost anything with remarkably little effort.

'USE ALL THE DATA'

There are differences between the way a student goes about solving problems and the way that a research mathematician does. One of Pólya's pieces of advice to students (and a good one) is: 'Did you use all the data?' Does the statement of the problem involve information that you haven't used?

This particular strategy is not so much about the mathematics as about what went on inside the teacher's or examiner's head. I recall W. W. Sawyer's graphic description in *Prelude to Mathematics* of the process of 'reconstructing the examiner'. The problem must have come from somewhere; somebody thought it up. How? Often that very question suggests a line of attack.

But at research level *you invent the problem yourself*. It doesn't come to you with set hypotheses and a set conclusion, nor is there any guarantee that there is an answer at all. You therefore spend a great deal of time developing a feel for the problem, trying to decide what the essential ideas and concepts should be and how everything fits together. This is, if you like, the planning phase, but it's often not very structured. You can use Pólya's strategy only when you've got a pretty clear idea of what you want to prove. 'Hang on, we haven't used the fact that *M* is seven-dimensional! Something isn't right . . .' You need to develop the ability to select the relevant information.

In order to be able to select the crucial information, and make use of it, mathematicians spend a great deal of their time acquiring both a broad background and a repertoire of more specific tricks. In the education of mathematicians it is important not to concentrate solely on problem-solving methods: the actual mathematical content is important too. And a lot of that content must be taught in a fairly conventional way. Life is too short for students to 'discover' by their own 'investigations' everything that they ought to know. I'm very happy that they should learn to work out ideas by themselves, but I'm also rather worried that, unless we're careful, the next generation of students will be able to talk the hind leg off a donkey about the thought-processes involved in solving problems but will be able to operate only at a very low level in terms of content. And I also feel there's an inherent contradiction in an 'investigation' whose success is measured by whether or not investigators find what the teacher wants them to find. There is a real danger here, and the key to avoiding it is to see problem-solving as a practical tool, part of the mathematician's mental equipment, but not as an end in itself.

Pólya knew all this. He intended the advice in his book to be applied flexibly, informed by intelligence and experience. In a sense it is advice to the teacher, who is assumed to recognize familiar sensations, rather than to the student. It is not, and never was meant to be, an easy method for forcing mathematical ability into human heads. I don't believe there is any such method: the ability comes from within those heads, and it can be nurtured and developed but not forced.

HEURISTICS IN THE KITCHEN

Let me give you an example of a problem that is very easy to solve if the 'right' background intuition has been developed but much harder if it has not. Many of you will have seen this particular problem before; if you haven't, you might like to experiment, using Pólya's heuristics, and see whether they work for you.

In 1988 I was involved in the preparation of a series of TV programmes whose aim was to convey the enjoyment of mathematics at an elementary level to a broad audience (it averaged 8 million over the seven-show series). The chosen vehicle was puzzles, solved by individuals or teams in the studio. On the surface it resembled just another game show, but the production team made a big effort to include significant mathematical ideas. Part of my job was to make sure they didn't overstep the bounds of accuracy in the effort to keep the ideas simple. We spent many days thinking about the thought processes involved in solving puzzles: they are very similar to those involved in solving a mathematical problem.

It became clear early on that mathematicians have at their disposal certain reflexes that are not naturally present in most of us – not so much techniques as points of view. A sense of symmetry is one, an ability to discard irrelevant information another. In some ways puzzles are more accurate models for research mathematics than problems in maths exams or textbooks are: in puzzles and research much of the information available has no bearing on the final answer, and the trick is to filter out the noise.

One puzzle was about connecting up items of kitchen equipment to electric plugs: A to a, B to b, C to c. The cables must not cross.

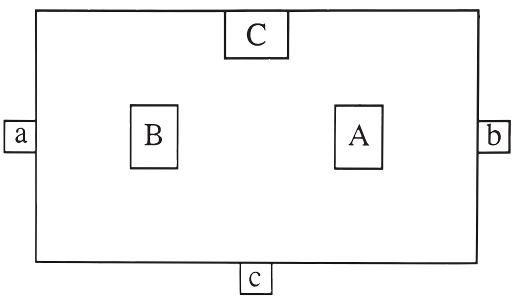

Figure 1

The line of argument – the heuristic – that we extracted from our deliberations is the following. Item C differs from the other two in that it is attached to a wall. So cable Cc cuts the kitchen in two, whereas Aa and Bb don't. If we run cable Cc the wrong way, we can cut off A from a and make our task impossible, like this:

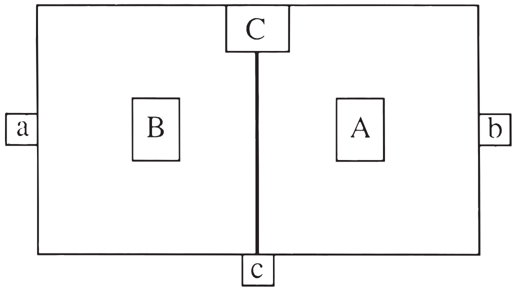

Figure 2

The way the puzzle is set deliberately deceives you into wanting to draw this incorrect connection. That's the art of puzzle-setting. A well-developed mathematical nose can sniff out these red herrings. The key mathematical idea here is *connectedness*. This is fundamentally a topological concept, and at a deeper level the puzzle is really about topology, but we felt that topology is too complicated for prime-time TV. However, I reckon that my audience here can tolerate a lot more, and I'll return to the topological idea below.

Research mathematicians know a very useful heuristic principle: *leave the hard bit to last*. Maybe you can knock off enough easy bits to find out that the problem isn't as hard as you'd thought. There's an equally valid principle that contradicts this completely: *go for the jugular*. It seems not to work as well on this particular puzzle, but sometimes it's better. One of the nice things about heuristics is that they don't have to be consistent. If one approach doesn't work, try another. You're in trouble only when you've run out of things to try.

Anyway, this suggests connecting up Aa and Bb first. How? However you please. Keep it simple:

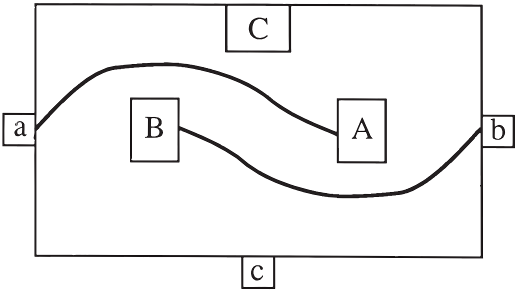

Figure 3

Now you've got a little maze to thread, and the answer's easy:

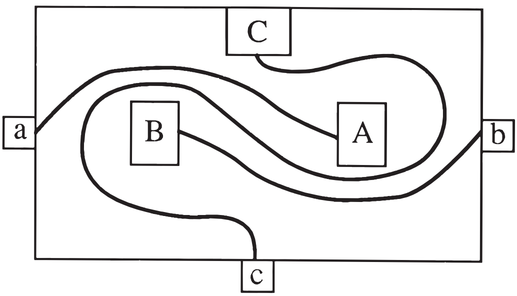

Figure 4

Great!

However, that's not how trained mathematicians will think about it. They have well-developed reflexes that come into play *immediately.* To the professional the puzzle is manifestly an exercise in topology. The shape of the room doesn't matter: a circle or ellipse would do just as well. In fact, you can distort the room any way you choose, as long as you do so by a *continuous* deformation. Not only the outline: *you can distort the floor too.*

In particular you can distort it so that A and B change places:

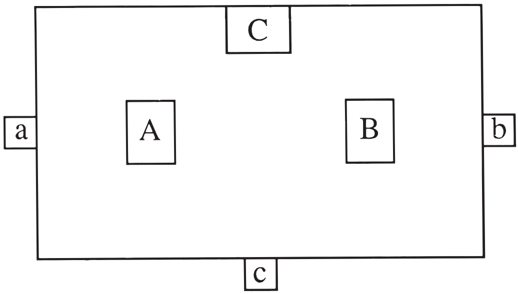

Figure 5

If the problem had been posed in this form to begin with, you'd have considered it a pretty awful puzzle because the answer stares you in the face:

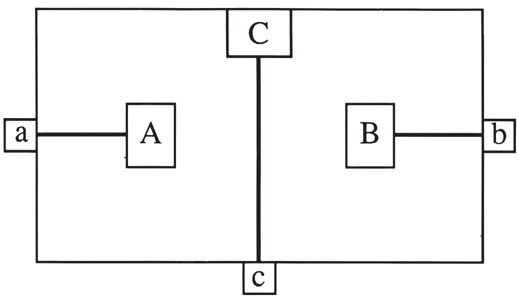

Figure 6

THE COUP DE GRÂCE

Except for one final thing: you have to recover the answer to the problem in its original form. Mathematicians are often so pleased that they've 'solved it' that they omit to write down this final step. They forget that what they've solved is some other problem that they happen to know is equivalent to the original one. The proof then comes to an abrupt and premature halt. This is a very bad habit because it confuses students no end. It's also a very difficult habit to shed. You stop when you *think* you've reached the end – that is, when the problem rolls over on its back and waves a white flag. When Caesar's given your opponent the thumbs-down, it's easy to forget you still have to administer the *coup de grâce*.

To do that in this case you've got to work out what the deformation that interchanges A and B actually does. Then you can undo it and carry the whole layout of the cables with you. Imagine the kitchen floor is made of putty. Stick two fingers into A and B, and twist your hand through 180 degrees, pulling the putty with you. Everything swirls round, continuously, and A and B change places. To undo this deformation you twist back the other way. You should now be able to see that twisting Figure 6 like this leads directly to Figure 4, the answer we wanted.

This reflex of the professional is not at all obvious. It has to be learnt, and it can be learnt only in a topology course. So what problem-solving resources you have at your disposal depends upon mathematical content as well as problem-solving strategy. New kinds of problem can lead to new strategies; new strategies can solve new kinds of problem. In fact, this is one of the most important ways in which really new mathematics gets developed.

But what, to my mind, is the real lesson that this puzzle teaches us is something that most of you will have to take on trust – though I assure you it's true. Most of us find Figure 1 quite hard and Figure 3 trivial and don't see any connection between the two. Topologists, by contrast, find it very hard to distinguish the two versions at all. *For topologists, the puzzle doesn't exist.*

Does the 'swirling' argument flit through their brain so fast that they don't consciously notice it? I don't think so. I think they just know immediately that it can be done. If pushed, they then think for a few seconds and come up with the 'swirling' image. I'm no topologist, but that's the order in which I 'solved' the puzzle. First I saw that it had to be trivial, then I had to rationalize why.

A student of topology may 'know' the theorem that any finite set of points in a rectangle can be moved to any positions you please by a topological transformation – and thus 'know' how to solve the puzzle – without being able to lay hands on the actual transformation required. There's a big difference between knowing facts and knowing how those facts fit together. There's also a difficult technical problem for the professional: to prove that what is intuitively obvious here is actually true. A logically rigorous proof that it is possible to interchange A and B by a topological transformation is surprisingly subtle. But all of this is part and parcel of a fully fledged topologist's mode of thought, and it is available immediately and without conscious effort.

If you ever get the chance, listen to mathematicians discussing research over coffee or lunch. Yes, they do that a lot – it's almost impossible to stop them (research, that is, not lunch). Anyway, you'll observe that they jabber incomprehensibly and wave their hands about. The two most important items of mathematical equipment are pen and paper. (And, as the old story has it, a wastepaper basket, thus distinguishing mathematics from philosophy.) It seems to be an almost unfailing rule that in any gathering of mathematicians nobody has any paper, so they use an old napkin. Nobody has a pen either, but they borrow one from somebody in the business studies department. What gets written on the napkin is cryptic and incoherent. Then suddenly one of them says, 'Oh! that's done it!' The others nod sagely. *What's done what?* you think. *How do they know?* But they've all simultaneously come to the same conclusion: they've seen the light at the end of the tunnel. From that moment they all agree that the problem is 'solved', even if working out the answer is a twenty-page calculation. The main idea has arrived. The problem has cracked wide open, ripe for plunder.

WEAK POINTS

Mathematical problems aren't uniformly impenetrable. They have their weak points, places where you can insert a probe, waggle it, exert some leverage, chip a bit off. Or, sometimes, crack the thing wide open.